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Algebra 1

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Problem Set #3

1 Modular arithmetic

Exercise 1 :
Check that $gcd(k, n) = 1$ and find $[k]^{-1}$ in $\mathbb{Z}/n\mathbb{Z}$ when $k = 296$, $n = 1317$.

Exercise 2 :
Determine $[a]^{-1}$ for each of the multiplicative units $[a] = [1], [5], [7], [11]$ in $\mathbb{Z}/12\mathbb{Z}$.

Exercise 3 :
Identify all element in $\mathbb{Z}/18\mathbb{Z}$ that have multiplicative inverse. Find $[5]^{-1}$ in this system by finding r, s such that $5r + 18s = 1$.

2 Rationals

Exercise 4 :
Prove that $\sqrt{3}$ is irrational.

3 Groups/Subgroups

Exercise 5 :
Which of the following set are groups ? (Explain your answer.)

1. $(\mathbb{Z}, \cdot)$;
2. $(\mathbb{R}, \cdot)$;
3. $((\mathbb{Z}/7\mathbb{Z})^{\times}, \cdot)$;

Exercise 6 :
Prove that

1. Knowing that $(\mathbb{Z}, +)$ is a group, prove that $(\mathbb{Z}/n\mathbb{Z}, \oplus)$ is a group ;
2. Knowing that $(\mathbb{R}, +)$ is a group, prove that $(\mathbb{R}^n, +)$ is a group ;

Exercise 7 :
Prove that

1. Prove that $(\Omega_n, \cdot)$ is a subgroup of $(\mathbb{C}^\times, \cdot)$, where $\Omega_n = \{z \in \mathbb{C} : z^n = 1\}$.
2. Prove that the orthogonal group $(O_n(\mathbb{R}) = \{M \in M_n(\mathbb{R}) : MM^T = I_n\}, \cdot)$ is a subgroup of $(GL_n(\mathbb{R}), \cdot)$.
3. Prove that the three-dimensional **Heisenberg group** of quantum mechanics consists of all real 3×3 matrices of the form

$$A = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with $x, y, z \in \mathbb{R}$ forms a subgroup of $(GL_n(\mathbb{R}), \cdot)$.
4. Prove that if $(G, \cdot)$ is a group and $S \subset G$ non empty subset,
   (a) $Z(G) = \{x \in G : gx = xg \text{ for all } g \in G\}$ is a subgroup of G ;
   (b) $Z_G(S) = \{x \in G : xs = sx \text{ for all } s \in S\}$ is a subgroup of G ;
   (c) $N_G(S) = \{x \in G : xSx^{-1} = S\}$ is a subgroup of G.
   (d) If H_α $(\alpha \in I)$ are subgroups of G, prove $H = \cap_{\alpha \in I} H_\alpha$ is also a subgroup.
5. Suppose $\phi : (G, \cdot) \to (G', *)$ is a homomorphism of groups, (e identity element of G and e' identity element of G'), prove that
   (a)
   $$\mathrm{Ker}(\phi) = \{x \in G : \phi(x) = e'\} \quad ,$$
   is a subgroup of G
   (b)
   $$\mathrm{Range}(\phi) = \phi(G) = \{\phi(x) : x \in G\}$$
   is a subgroup of G'.

Exercise 8 :
Evaluate the net action of the following product of cycles :
1. $(1, 2)(1, 3)$ in S_3 ;
2. $(1, 2)(1, 3)$ in S_5 ;
3. $(1, 5)(1, 4)(1, 3)(1, 2)$ in S_5 ;

Exercise 9 :
Find the inverses σ^{-1} in S_5 :
1. $(1, 2)$;
2. $(1, 2, 3)$;
3. For any cycle (i_1, i_2) with $i_1 \neq i_2$;
4. $(i_1, i_2, \ldots, i_k)$ with $i_k \neq i_l$ for $k \neq l$.